As will be seen from the following small investigation, the inequality of the problem holds if and only if the triangle ABC is not obtuse.

The small investigation about this problem.

https://www.linkedin.com/feed/update/urn:li:activity:6601685305533837312

In triangle ABC, if L, M, N are midpoints of AB, AC, BC. And H is orthocenter of triangle ABC, then prove that $LH^2 + MH^2 + NH^2 \leq (1/4)(AB^2 + AC^2 + BC^2)$.

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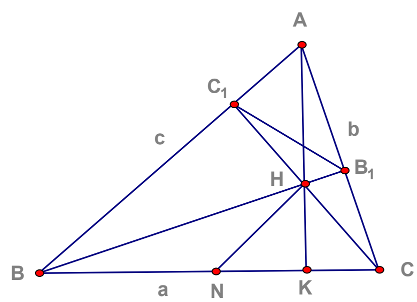

Let R be circumradius of $\triangle ABC$. Since $AK = b\sin C = 2R\sin B \cdot \sin C$, $AH = 2R\cos A = 2R\cos A$ (because $\triangle A\acute{B}_1C_1 \sim \triangle ABC$ with coefficient of similarity $\cos A$) then $HK = AK - AH = 2R\sin B\sin C - 2R\cos A =$ $2R(\sin B \cdot \sin C + \cos(B + C)) = 2R\cos B \cdot \cos C$. Noting that $KC = b\cos C$ we obtain $NK = |NC - CK| = \left|\frac{a}{2} - b\cos C\right| = \left|\frac{c\cos B + b\cos C}{2} - b\cos C\right| =$ $\left|\frac{c\cos B - b\cos C}{2}\right| = R|\sin C \cdot \cos B - \sin B\cos C| = R|\sin(B - C)|$ we obtain

$NH^2 = NK^2 + HK^2 = R^2\left((\sin C \cdot \cos B - \sin B \cdot \cos C)^2 + 4\cos^2 B \cdot \cos^2 C\right) =$

$R^2(\sin^2 C \cdot \cos^2 B + \sin^2 B \cdot \cos^2 C - 2\sin C \cdot \cos B \cdot \sin B \cdot \cos C + 4\cos^2 B \cdot \cos^2 C) =$

$R^2(\cos^2 B(\sin^2 C + \cos^2 C) + \cos^2 C(\sin^2 B + \cos^2 B) + 2\cos B\cos C \cdot \cos(B + C)) =$

$R^2(\cos^2 B + \cos^2 C - 2\cos A\cos B\cos C)$.

Hence, $LH^2 + MH^2 + NH^2 = \sum R^2(\cos^2 B + \cos^2 C + 2\cos A\cos B\cos C) =$ $R^2(2(\cos^2 A + \cos^2 B + \cos^2 C) - 6\cos A\cos B\cos C)$ and since $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1$ (in any triangle ABC) then $LH^2 + MH^2 + NH^2 = R^2(5(\cos^2 A + \cos^2 B + \cos^2 C) - 3) =$ $R^2(12 - 5(\sin^2 A + \sin^2 B + \sin^2 C)) = 12R^2 - \frac{5}{4}(a^2 + b^2 + c^2)$

and now we ready to promised investigation.

Since in any triangle holds inequality $a^2 + b^2 + c^2 \leq 9R^2$ then $12R^2 - \frac{5}{4}(a^2 + b^2 + c^2) \geq 12R^2 - \frac{5}{4} \cdot 9R^2 = \frac{3}{4}R^2$ that is in any triangle holds inequality $LH^2 + MH^2 + NH^2 \geq \frac{3}{4}R^2$.

If triangle ABC is not acute then $a^2 + b^2 + c^2 \leq 8R^2$ and, therefore

$12R^2 - \frac{5}{4}(a^2 + b^2 + c^2) \geq 12 \cdot \frac{a^2 + b^2 + c^2}{8} - \frac{5}{4}(a^2 + b^2 + c^2) = \frac{1}{4}(a^2 + b^2 + c^2)$,

that is $LH^2 + MH^2 + NH^2 \geq \frac{1}{4}(a^2 + b^2 + c^2)$ in any not acute triangle.

And at last, if $\triangle ABC$ isn't obtuse then $a^2 + b^2 + c^2 \leq 8R^2$ and, therefore

$$12R^2 - \frac{5}{4}(a^2 + b^2 + c^2) \leq 12 \cdot \frac{a^2 + b^2 + c^2}{8} - \frac{5}{4}(a^2 + b^2 + c^2) = \frac{1}{4}(a^2 + b^2 + c^2)$$

that is $LH^2 + MH^2 + NH^2 \leq \frac{1}{4}(a^2 + b^2 + c^2)$ in any not obtuse triangle

And in the both latter inequalities equality occurs iff triangle ABC is right angled.

Thus, inequality $LH^2 + MH^2 + NH^2 \leq (1/4)(AB^2 + AC^2 + BC^2)$ holds iff $\triangle ABC$ isn't obtuse.